Туршиянц 202, (739)

(1).

$$\vec{a}=\vec{r}r^2=\{x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2)^2\}.$$

$$\iint_S (\vec{a},\vec{n})\,ds=\iiint_V \operatorname{div}\vec{a}\,dV$$

$$\operatorname{div}\vec{a}=\frac{\partial(x(x^2+y^2+z^2))}{\partial x}+\frac{\partial(y(x^2+y^2+z^2))}{\partial y}+\frac{\partial(z(x^2+y^2+z^2))}{\partial z}=$$

$$=\cancel{x^2+y^2+z^2+2x^2+2y^2+2z^2}=$$

$$=3(x^2+y^2+z^2)+2(x^2+y^2+z^2)=5(x^2+y^2+z^2).$$

$$\iiint_V 5(x^2+y^2+z^2)\,dx\,dy\,dz=\begin{cases}x=r\cos\varphi\sin\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\theta\\ J=r^2\sin\theta\end{cases}=$$

$$=5\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^{2\pi}d\varphi\int_0^1 r^4\,dr=2\pi.$$

(3).

$$\operatorname{div}(r^2\vec{r})=5(x^2+y^2+z^2).$$

$$\operatorname{grad}(5(x^2+y^2+z^2))=\{10x,10y,10z\}=10\vec{r}.$$

(2). $\operatorname{div}(\vec{a}(\rho,\varphi,\theta)).$

$$\begin{cases}x=\rho\sin\theta\cos\varphi\\ y=\rho\sin\theta\sin\varphi\\ z=\rho\cos\theta\end{cases},\quad \begin{matrix}0\le\rho\le+\infty\\ 0\le\varphi\le2\pi\\ 0\le\theta\le\pi\end{matrix}\quad (\rho,\varphi,\theta)=(x^1,x^2,x^3).$$

$$r_1=\{\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta\},\ H_1=|r_1|=1$$

$$r_2=\{-\rho\sin\theta\sin\varphi,\rho\sin\theta\cos\varphi,0\},\ H_2=|r_2|=\rho\sin\theta$$

$$r_3=\{\rho\cos\theta\cos\varphi,\rho\cos\theta\sin\varphi,-\rho\sin\theta\},\ H_3=|r_3|=\rho.$$

$$r_1 r_2 r_3 = \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\rho\sin\theta\sin\varphi & \rho\sin\theta\cos\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\cos\theta\sin\varphi & -\rho\sin\theta \end{vmatrix} = -\rho^2\sin\theta.$$

4)

$$r^1 = \frac{[r_2 r_3]}{r_1 r_2 r_3} = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\}$$

$$r^2 = \frac{(r_3 r_1)}{r_1 r_2 r_3} = \left\{-\frac{\sin\varphi}{\rho\sin\theta}, \frac{-\cos\varphi}{\rho\sin\theta}, 0\right\}$$

$$r^3 = \frac{(r_1 r_2)}{r_1 r_2 r_3} = \left\{+\frac{\cos\theta\cos\varphi}{\rho}, \frac{\cos\theta\sin\varphi}{\rho}, -\frac{\sin\theta}{\rho}\right\}.$$

$$\operatorname{div}\vec{a} = \operatorname{div}A = r^i A r_i, \quad \frac{\partial a}{\partial q^i} = a_i.$$

$$\operatorname{div}\vec{a} = r^i a_i \Rightarrow \operatorname{div}\vec{a} = \frac{a_1 r_1}{H_1^2} + \frac{a_2 r_2}{H_2^2} + \frac{a_3 r_3}{H_3^2}$$

$$\Rightarrow \operatorname{div}\vec{a} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2 A_\rho\right) + \frac{1}{\rho\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} + \frac{1}{\rho\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta}$$

5) $\oint_C([\vec{r}\times\vec{a}], d\vec{l})$

$$\vec{b} = [\vec{r}\times\vec{a}] = \begin{vmatrix} i & j & k \\ x & y & z \\ a_x & a_y & a_z \end{vmatrix} = \vec{i}(a_z y - a_y z) + \vec{j}(a_x z - a_z x) + \vec{k}(a_y x - a_x y).$$

$$\operatorname{rot}\vec{b} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_z y - a_y z & a_x z - a_z x & a_y x - a_x y \end{vmatrix} = \vec{i}(-a_x - a_x) + \vec{j}(-a_y - a_y) + \vec{k}(-a_z - a_z) = -2\vec{a}.$$

$$\oint_c ([\vec{u}\times\vec{a}], d\vec{l}) = \iint_S (\mathrm{rot}\,\vec{b}, \vec{n})\, ds = \cancel{\ldots} = \cancel{\ldots} =$$

$$= -2(\vec{a},\vec{n}) \iint_S ds = -2(\vec{a},\vec{n}) \cdot \pi r^2 \overset{1}{=} -2\pi(\vec{a},\vec{n}) =$$

$$= 2\pi \cdot a_z.$$

4.

$$\mathrm{rot}(u(\vec{r})) = \vec{i}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \vec{j}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \vec{k}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial z}\right)$$

$$(u(\vec{r}), \mathrm{rot}(u(\vec{r}))) = \vec{i}\, u_x\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \vec{j}\left(u_y\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial x}\right)\right) +$$

$$+ \vec{k}\, u_z\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right).$$

$$\Rightarrow \mathrm{div}(\ \ ) = \frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial y} - \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_x}{\partial z} - \frac{\partial u_y \partial u_z}{\partial y \partial x} +$$

$$+ \frac{\partial u_z \partial u_y}{\partial z \partial x} - \frac{\partial u_z \partial u_x}{\partial z \partial y} = u_x \frac{\partial u_z^2}{\partial y \partial x} - u_x \frac{\partial u_y^2}{\partial z \partial x} + u_y \frac{\partial u_x^2}{\partial z \partial y} -$$

$$- u_y \frac{\partial u_z^2}{\partial x \partial y} + u_z \frac{\partial u_y^2}{\partial x \partial z} - u_z \frac{\partial u_x^2}{\partial y \partial z}$$